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# Exact frequency equations of free vibration of exponentially functionally graded beams

X.-F. Li *, Y.-A. Kang, J.-X. Wu

School of Civil Engineering, Central South University, Changsha 410075, China

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ABSTRACT

Free vibration of axially inhomogeneous beams is analyzed. For exponentially graded beams with various end conditions, characteristic equations are derived in closed form. These characteristic or frequency equations can analytically reduce to the classical forms of Euler–Bernoulli beams if the gradient index disappears. The gradient has a strong influence on the frequency spectrum, and the natural frequencies noticeably depend on the variation of the gradient parameter and end support conditions. For certain beams with exponential gradients, there exists a critical frequency depending on the gradient parameter. Vibration can be only excited by propagating waves with frequencies in excess of the critical frequency, and otherwise vibration is prohibited for pseudo-frequencies lower than the critical frequency. For some gradient index with small change, the natural frequencies have an abrupt jump when across its critical frequencies. Obtained results can serve as a benchmark for other numerical procedures for analyzing transverse vibration of axially functionally graded beams. The minimal natural frequency can be sought for certain gradient index, and this helps engineers to optimally design vibrating nonhomogeneous beam structures. Obtained results also apply to free vibration of nonuniform beams with constant thickness and exponentially decaying width.



## 1. Introduction

Functionally graded beams have attracted increasing attention of researchers in recent years. The material properties of functionally graded beams vary along the thickness direction or/and the length direction. For functionally graded beams with thickness-wise gradient variation, great progress has been made. For example, Sankar [1] made a static analysis of thickness-wise exponentially graded beams with pinned–pinned ends according to two-dimensional elasticity model and one-dimensional Euler–Bernoulli beam model. Zhong and Yu [2] formulated an analytical solution of a static cantilever functionally graded beam with the assumption that all the elastic moduli of the material have the same variations along the beam-thickness direction. Later, the static analysis was further extended to anisotropic functionally graded beams by Ding et al. [3]. Using the discrete quadrature method, Lu et al. [4] presented a semi-analytical elasticity solution for static problems of bidirectional functionally graded beams with exponential gradient distribution within the framework of two-dimensional elasticity theory. For the case of power-law nonlinear constitutive relations, the exact deflection of transverse bending of cantilevered functionally graded beams under small and large deformation has been obtained under different applied loads [5,6]. In certain situations, dynamic analysis of functionally graded beams is of much interest. Aydogdu and Taskin [7] investigated free vibration of simply-supported functionally graded beams where Young's modulus vary in the thickness direction according to power law and exponential law. When neglecting axial force and corresponding longitudinal stretching, Li [8] presented a new unified approach for analyzing transverse bending and dynamic behaviors of functionally graded beams with rotary inertia and shear deformation included. Sina et al. [9] also used Hamilton's principle to obtain the natural frequencies and mode shapes of functionally graded beams. Simsek [10] employed different higher-order beam theories to compute the fundamental frequencies. For a functionally graded beam with circular cross-section of arbitrary radial gradient, Huang and Li [11] established a high-order theory of beams including shear deformation and rotary inertia where traction-free surface condition is identically met. Based on a three-dimensional theory, Kang and Leissa [12] gave a vibration analysis of thick, tapered rods and beams with circular cross-section.

For functionally graded beams with axially varying material properties, related researches are quite limited. Elishakoff and co-workers have made a great deal of work on finding the exact solutions of fundamental frequency for a majority of beams with various end supports via using the semi-inverse method [13]. The semi-inverse method can give a closed-form solution, but it

* Corresponding author. Tel.: +86 731 8816 7070; fax: +86 731 8557 1736.
E-mail address: xfli@mail.csu.edu.cn (X.-F. Li).

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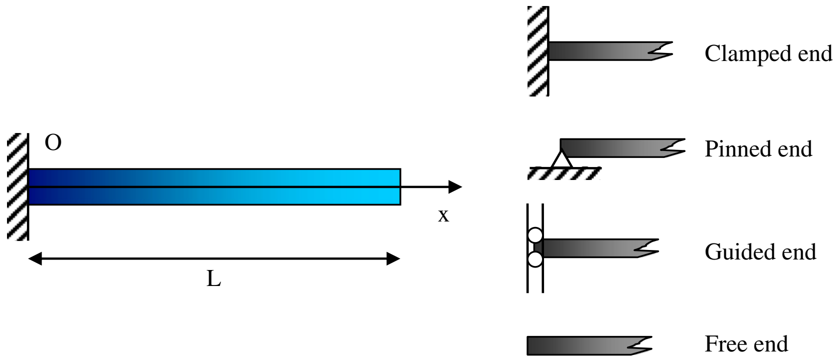


**Fig. 1.** Schematic of an axially functionally graded beam.

is only applicable for specific bending stiffness and mass density like some special polynomials. In particular, it is inconvenient for determining higher-order natural frequencies. Huang and Li [14] studied free vibration of axially graded beams with any gradient variation by using the integral equation method. It is worth noting that the proposed method for seeking the natural frequencies is approximate, not exact. Murin et al. [15] formulated a method for dealing with the natural frequencies of beams of varying material properties. Shahba et al. [16] coped with free vibration and stability analysis of axially functionally graded tapered Timoshenko beams with classical and non-classical boundary conditions via finite element method. Li et al. [17] gave an optimization of axially graded beams against buckling. Chakrabarti et al. [18] studied large amplitude free vibration of a rotating nonhomogeneous beam with nonlinear spring and mass system.

As we know, in many cases, analytical solutions in closed-form are desired for accurate analysis and design due to their many advantages over numerical and approximate solutions. Moreover, analytical solutions can serve as a benchmark for the purpose of judging the accuracy and efficiency of various numerical and approximate approaches. However, owing to the mathematical complexity, certain practical problems are only solved with recourse to numerical and approximate schemes.

This paper presents an analytic treatment of the free vibration of axially exponentially graded beams. The frequency equations of beams with various end conditions are derived in closed form. For inhomogeneous beams, we find an essential difference from homogeneous beams. For certain beams with exponential gradient, vibration of the beams cannot take place in frequencies lower than a critical value. Only when frequencies exceed the critical value, harmonic vibration occurs.

## 2. Basic equations

Consider a beam of varying cross section or made of axially graded materials. The beam length is denoted as $L$, and the axial direction is denoted as $x$ with the origin at the end $x = 0$, as shown in Fig. 1. The bending stiffness and the distributed mass are also dependent on the axial coordinate $x$. For simplicity, we assume

$$EI = De^{2\beta x/L}, \quad \rho A = me^{2\beta x/L}, \tag{1}$$

where $\beta$ is the dimensionless gradient parameter, $EI$ is varying bending stiffness and $D$ is a reference value of $EI$ at $x = 0$, $\rho A$ is distributed mass per unit length and $m$ is a reference value of $\rho A$ at $x = 0$. Here $I$ is the second-order moment of cross-sectional area, $A$ the cross-sectional area, $E$ Young's modulus, and $\rho$ the mass density, respectively. Notice that as $x$ is raised, $\beta > 0$ indicates that $EI$ and $\rho A$ increase, and $\beta < 0$ indicates $EI$ and $\rho A$ decrease, respectively.

For a beam with varying cross section, the above assumption corresponds to the cross section possessing exponentially varying width and constant thickness, while for an axially graded beam, the assumption means that the material properties have exponential gradient. For such beams with material properties described by (1), the differential equation that governs the transverse vibration of the beams reads

$$\frac{\partial^2}{\partial x^2}\left[De^{2\beta x/L}\frac{\partial^2 w}{\partial x^2}\right] + me^{2\beta x/L}\frac{\partial^2 w}{\partial t^2} = 0, \quad 0 < x < L. \tag{2}$$

When the beam undergoes transverse vibration, one may take $w$ in the form

$$w(x,t) = W(x)\sin\omega t, \tag{3}$$

where $W(x)$ is the amplitude of vibration, and $\omega$ is the circular frequency of vibration. After plugging the above expression into (2), one gets

$$\frac{\partial^2}{\partial x^2}\left[De^{2\beta x/L}\frac{\partial^2 W}{\partial x^2}\right] - m\omega^2 L^4 e^{2\beta x/L} W = 0. \tag{4}$$

Hereafter, we still use notation $\partial()/\partial x$ to stand for $d()/dx$ without confusion.

In the present paper, emphasis is placed on the effect of the gradient index $\beta$ on the natural frequencies of the transverse vibration. As will be seen below, some new features can be found in the light of the appearance of gradient variation. For convenience of later analysis let us introduce dimensionless quantities

$$\xi = \frac{x}{L}, \quad \Omega = \omega L^2\sqrt{\frac{m}{D}}. \tag{5}$$

Thus Eq. (4) can be rewritten as

$$\frac{\partial^2}{\partial \xi^2}\left[e^{2\beta\xi}\frac{\partial^2 W}{\partial \xi^2}\right] - \Omega^2 e^{2\beta\xi} W = 0. \tag{6}$$

With $W(x)$ at hand, the bending moment $M$ and shear force $Q$ at any cross section are given respectively in terms of the introduced dimensionless variables by

$$M = -\frac{De^{2\beta\xi}}{L^2}\frac{\partial^2 W}{\partial \xi^2}, \tag{7}$$

$$Q = -\frac{\partial}{\partial \xi}\left(\frac{De^{2\beta\xi}}{L^3}\frac{\partial^2 W}{\partial \xi^2}\right). \tag{8}$$

## 3. Free vibration

In order to obtain a nontrivial solution of the above Eq. (6), it is expedient to suppose that $W$ takes the form

$$W(\xi) = Ce^{\lambda\xi}, \tag{9}$$

where $\lambda$ and $C$ are two unknown constants. After substituting the above expression for $W$ into (6), we get

$$\left[\lambda^2(\lambda + 2\beta)^2 - \Omega^2\right]C = 0. \tag{10}$$

Due to $C$ being a non-vanishing constant, we get $\lambda^2(\lambda+2\beta)^2 - \Omega^2 = 0$, and then four possible roots, called characteristic values, can be easily obtained by directly solving the above equation, i.e.

$$\lambda_{1,3} = -\beta \pm i\sqrt{\Omega - \beta^2}, \quad \lambda_{2,4} = -\beta \pm \sqrt{\beta^2 + \Omega}, \text{ if } \Omega > \beta^2, \tag{11}$$

or

$$\lambda_{1,3} = -\beta \pm \sqrt{\beta^2 - \Omega}, \quad \lambda_{2,4} = -\beta \pm \sqrt{\beta^2 + \Omega}, \text{ if } \Omega \leqslant \beta^2. \tag{12}$$

If denoting

$$\delta_1 = \begin{cases} \sqrt{\Omega - \beta^2}, & \text{if } \Omega > \beta^2, \\ 0, & \text{if } \Omega = \beta^2, \\ \sqrt{\beta^2 - \Omega}, & \text{if } \Omega < \beta^2, \end{cases} \quad \delta_2 = \sqrt{\Omega + \beta^2}, \tag{13}$$

we obtain possible expressions for the mode shape taking the following form for individual cases

$$W(\xi) = e^{-\beta\xi}[C_1\cos(\delta_1\xi) + C_2\sin(\delta_1\xi) + C_3\cosh(\delta_2\xi) + C_4\sinh(\delta_2\xi)], \quad \text{if } \Omega > \beta^2, \tag{14}$$

$$W(\xi) = e^{-\beta\xi}[C_1 + C_2\xi + C_3\cosh(\delta_2\xi) + C_4\sinh(\delta_2\xi)], \quad \text{if } \Omega = \beta^2, \tag{15}$$

$$W(\xi) = e^{-\beta\xi}[C_1\cosh(\delta_1\xi) + C_2\sinh(\delta_1\xi) + C_3\cosh(\delta_2\xi) + C_4\sinh(\delta_2\xi)], \quad \text{if } \Omega < \beta^2. \tag{16}$$

From the above, an important conclusion can be implied. That is, if frequencies are lower enough such that $\Omega \leqslant \beta^2$, we find that obtained four roots are all real. This indicates that the mode shape of vibration $W$ is not harmonic. In other words, such vibration modes given by (15) and (16) do not correspond to propagating waves, but represent nonpropagating fields or evanescent components. Therefore, according to the usual convention, such values have no realistic physical meanings. The lowest value, $\beta^2$, is referred to as the dimensionless critical frequency $\Omega_{cr} = \beta^2$. Once the frequency exceeds the critical value, propagating waves can be excited and harmonic vibration is then induced. On contrary, if the frequency is less than the critical frequency, propagating waves cannot be excited and harmonic vibration of the beams is impossible. This is easily understood since for the former, the corresponding characteristic vector is a trigonometric function which must vanish at one or more certain positions at the beam. For the latter case, nevertheless, the corresponding characteristic vector never vanishes at any position of the beam since it is an exponential function related to a nonpropagating fields or evanescent components although their combination may vanish at certain positions like end(s). This is an essential difference between propagating and nonpropagating waves when the frequency crosses the critical frequency. Although some characteristic values of $\Omega$ lower than or equal to $\beta^2$ can be calculated, due to the reason stated above these characteristic values are called pseudo-frequencies. It is worth noting that a theoretical analysis of exponential mode of vibration of beams can be made [19]. Nevertheless, in what follows we only restrict our attention to the case where the frequencies are above the critical frequency, meaning that the characteristic values are given by (11) or equivalently, harmonic vibration may be excited.

Next, we consider several typical free vibration of exponentially graded beams. To this end, we express the amplitude $W$, rotation $\partial W/\partial\xi$, bending moment $M$, and shear force $Q$ in terms of the dimensionless variables

$$W = e^{-\beta\xi}[A_1, A_2, A_3, A_4]\mathbf{V}, \tag{17}$$

$$\frac{\partial W}{\partial\xi} = e^{-\beta\xi}[\delta_1A_2 - \beta A_1, -(\delta_1A_1 + \beta A_2), \delta_2A_4 - \beta A_3, \delta_2A_3 - \beta A_4]\mathbf{V}, \tag{18}$$

$$M = -\frac{De^{\beta\xi}}{L^2}\big[-2\beta\delta_1A_2 + (\beta^2 - \delta_1^2)A_1, 2\beta\delta_1A_1 + (\beta^2 - \delta_1^2)A_2, -2\beta\delta_2A_4 + (\beta^2 + \delta_2^2)A_3, -2\beta\delta_2A_3 + (\beta^2 + \delta_2^2)A_4\big]\mathbf{V}, \tag{19}$$

$$Q = -\frac{\Omega^2De^{\beta\xi}}{L^3}[\beta A_1 - \delta_1A_2, \delta_1A_1 + \beta A_2, \delta_2A_4 - \beta A_3, \delta_2A_3 - \beta A_4]\mathbf{V}, \tag{20}$$

where $A_j$ and $B_j$ are non-vanishing constants, and

$$\mathbf{V} = [\cos\delta_1\xi, \sin\delta_1\xi, \cosh\delta_2\xi, \sinh\delta_2\xi]^T. \tag{21}$$

In the above, the superscript $T$ specifies the transpose of a vector or matrix.

To determine the natural frequencies of exponentially graded beams, four independent algebraic equations are needed. These equations can be furnished by using the boundary conditions at end supports.

## 4. Frequency equations

In this section, we will consider several typical end supports of a beam and derive their frequency equations. Furthermore, the natural frequencies are evaluated and mode shapes are given.

### 4.1. Pinned–pinned beams

Firstly, let us consider pinned–pinned beams or simply-supported beams and the associated boundary conditions read

$$W(0) = 0, \quad M(0) = 0, \tag{22}$$

$$W(1) = 0, \quad M(1) = 0. \tag{23}$$

Using these conditions, from (17) and (19) we get the following equations

$$A_1 + A_3 = 0, \tag{24}$$

$$-2\beta\delta_1A_2 + (\beta^2 - \delta_1^2)A_1 - 2\beta\delta_2A_4 + (\beta^2 + \delta_2^2)A_3 = 0, \tag{25}$$

$$[A_1, A_2, A_3, A_4]\mathbf{V}(1) = 0, \tag{26}$$

$$\big[-2\beta\delta_1A_2 + (\beta^2 - \delta_1^2)A_1, 2\beta\delta_1A_1 + (\beta^2 - \delta_1^2)A_2, -2\beta\delta_2A_4 + (\beta^2 + \delta_2^2)A_3, -2\beta\delta_2A_3 + (\beta^2 + \delta_2^2)A_4\big]\mathbf{V}(1) = 0, \tag{27}$$

where

$$\mathbf{V}(1) = [\cos\delta_1, \sin\delta_1, \cosh\delta_2, \sinh\delta_2]^T. \tag{28}$$

Hence, we obtain a system of linear equations in $A_j$ ($j$ = 1, 2, 3, 4). Since this system has a nontrivial solution, the determinant of the coefficient matrix of this system has to vanish, which finally simplifies to

$$2\beta^2\delta_1\delta_2(1 - \cosh\delta_2\cos\delta_1) + (2\beta^4 - \Omega^2)\sinh\delta_2\sin\delta_1 = 0. \tag{29}$$

This is an exact characteristic equation or frequency equation of pinned–pinned beams with exponentially varying bending stiffness and distributed mass (1). In the above equation, if setting $\beta = 0$ one further gets

$$\sin\sqrt{\Omega} = 0, \tag{30}$$

which is identical to the characteristic equation for simply-supported beams of uniform cross section and homogeneous materials [20], as expected.

### 4.2. Clamped–clamped beams

Here, let us turn our attention to clamped–clamped beams and the associated boundary conditions read

$$W(0) = 0, \quad \left.\frac{\partial W}{\partial \xi}\right|_{\xi=0} = 0, \tag{31}$$

$$W(1) = 0, \quad \left.\frac{\partial W}{\partial \xi}\right|_{\xi=1} = 0. \tag{32}$$

Using these conditions, from (17)–(20) we get the following equations

$$A_1 + A_3 = 0, \tag{33}$$

$$\delta_1 A_2 - \beta A_1 + \delta_2 A_4 - \beta A_3 = 0, \tag{34}$$

$$[A_1, A_2, A_3, A_4]\mathbf{V}(1) = 0, \tag{35}$$

$$[\delta_1 A_2 - \beta A_1, -(\delta_1 A_1 + \beta A_2), \delta_2 A_4 - \beta A_3, \delta_2 A_3 - \beta A_4]\mathbf{V}(1) = 0. \tag{36}$$

Hence, we also obtain a system of linear equations in $A_j$ ($j$ = 1, 2, 3, 4). Since this system has a nontrivial solution, the determinant of the coefficient matrix of this system has to vanish, which finally simplifies to

$$\delta_1\delta_2(1 - \cosh\delta_2\cos\delta_1) + \beta^2\sinh\delta_2\sin\delta_1 = 0. \tag{37}$$

This is an exact characteristic or frequency equation of clamped–clamped beams with exponentially varying bending stiffness and distributed mass (1). In the above equation, if setting $\beta = 0$ one further gets

$$1 - \cos\sqrt{\Omega}\cosh\sqrt{\Omega} = 0, \tag{38}$$

which is identical to the characteristic equation for clamped–clamped beams of uniform cross section and homogeneous beams [20], as expected.

### 4.3. Clamped–free beams

For clamped–free beams, the associated boundary conditions read

$$W(0) = 0, \quad \left.\frac{\partial W}{\partial \xi}\right|_{\xi=0} = 0, \tag{39}$$

$$M(1) = 0, \quad Q(1) = 0. \tag{40}$$

Using these conditions, from (17)–(20) we get the following equations

$$A_1 + A_3 = 0, \tag{41}$$

$$\delta_1 A_2 - \beta A_1 + \delta_2 A_4 - \beta A_3 = 0, \tag{42}$$

$$\begin{aligned}&[-2\beta\delta_1 A_2 + (\beta^2 - \delta_1^2)A_1, 2\beta\delta_1 A_1 + (\beta^2 - \delta_1^2)A_2, -2\beta\delta_2 A_4 \\ &+ (\beta^2 + \delta_2^2)A_3, -2\beta\delta_2 A_3 + (\beta^2 + \delta_2^2)A_4]\mathbf{V}(1) = 0,\end{aligned} \tag{43}$$

$$[\beta A_1 - \delta_1 A_2, \delta_1 A_1 + \beta A_2, \delta_2 A_4 - \beta A_3, \delta_2 A_3 - \beta A_4]\mathbf{V}(1) = 0. \tag{44}$$

Hence, we obtain a system of linear equations in $A_j$ ($j$ = 1, 2, 3, 4). Since this system has a nontrivial solution, the determinant of the coefficient matrix of this system has to vanish, which finally simplifies to

$$\begin{aligned}&\delta_1\delta_2 + \delta_2(\delta_1\cos\delta_1 - 2\beta\sin\delta_1)\cosh\delta_2 \\ &- \beta(2\delta_1\cos\delta_1 - 3\beta\sin\delta_1)\sinh\delta_2 = 0.\end{aligned} \tag{45}$$

This is an exact characteristic or frequency equation of the cantilevers with exponentially varying bending stiffness and distributed mass (1). In the above equation, if setting $\beta = 0$ one further gets

$$1 + \cos\sqrt{\Omega}\cosh\sqrt{\Omega} = 0, \tag{46}$$

which is identical to the characteristic equation for uniform homogeneous beams with clamped–free ends [20], as expected.

For other end conditions we can similarly derive frequency equations. For saving spaces, we omit the details of the derivation and the final results are listed in Table 1. It is worth noting that for nonuniform Euler–Bernoulli beams with exponential bending stiffness and mass density or equivalently with constant thickness and exponentially decaying width, the frequency equations for three cases: pinned–pinned beams, clamped–clamped beams, and clamped–free beams, have been derived in [21–24]. In particular, it is found that our frequency equation for clamped-free beams is inconsistent with previous works [23,24]. The reason is that the free end conditions in (40) we used here are different from those in [23]. Recently, Wang and Wang [25] treated the free vibration of a cantilever beam with constant thickness and exponentially decaying width carrying a tip mass. For other end supports, the frequency equations were not reported, to the best of the authors' knowledge. From Table 1, we can find that three pairs of the characteristic equations are the same. They are those for clamped–clamped beams versus free–free beams, clamped–guided beams versus guided–free beams, clamped–pinned beams versus pinned–free beams. If imposing $\beta = 0$, the characteristic equations in Table 1 reduce to those for uniform beams made of homogeneous materials [20].

## 5. Natural frequencies and mode shapes

Since the explicit characteristic equations have been presented in Table 1 for various frequently-encountered cases, it is a very simple matter to obtain numerical natural frequencies by solving the above-obtained characteristic equations with the aid of commercial software. However, it is particularly pointed out that for certain exponential gradient parameter $\beta$, calculated values of the dimensionless natural frequencies are possibly less than the dimensionless critical frequency $\Omega_{cr} = \beta^2$. This directly gives rise to the mode shapes being an exponential function or equivalently hyperbolic function rather than a trigonometric function, and it then does not correspond to harmonic vibration excited by propagating waves. As a consequence, the natural frequencies to be determined are always larger than the critical frequency.

Fig. 2 shows the effects of the gradient parameter $\beta$ on the natural frequency parameter $\sqrt{\Omega}$. In Fig. 2, the first four natural frequency parameter curves are plotted. From Fig. 2, it is found that all the frequency curves are discontinuous. For example, the fundamental frequency parameter $\sqrt{\Omega_1}$ displayed in solid lines is seen to decline with $\beta$ rising, reaching a minimum critical value about $\sqrt{\Omega_1} = 1.2$, and then jumps up to the curve above and next to this curve since the frequency computed from this curve disappears. This is attributed to the fact of $\Omega_1^2 < \beta$. Due to this reason, the original second natural frequency is then converted to the fundamental frequency. Similarly, with $\beta$ increasing, the frequency parameter $\sqrt{\Omega_1}$ further reduces to fall into the shadow region of $\Omega_1^2 < \beta$ again as $\beta$ is larger than 3.2. Since the fundamental

**Table 1**
Characteristic equations for various beams.

| Boundary conditions | Characteristic equations |
|---|---|
| C–C | $\delta_1\delta_2\,(1-\cosh\delta_2\cos\delta_1)+\beta^2\sinh\delta_2\sin\delta_1=0$ |
| P–P | $2\beta^2\delta_1\delta_2\,(1-\cosh\delta_2\cos\delta_1)+(2\beta^4-\Omega^2)\,\sinh\delta_2\sin\delta_1=0$ |
| G–G | $\sin\delta_1=0$ |
| F–F | $\delta_1\delta_2\,(1-\cosh\delta_2\cos\delta_1)+\beta^2\sinh\delta_2\sin\delta_1=0$ |
| C–F (or F–C) | $\delta_1\delta_2\,(1+\cosh\delta_2\cos\delta_1)-2\beta$ $(\delta_1\sinh\delta_2\cos\delta_1+\delta_2\cosh\delta_2\sin\delta_1)+3\beta^2\sinh\delta_2\sin\delta_1=0$ |
| C–P (or P–C) | $2\beta\delta_1\delta_2\,(1-\cosh\delta_2\cos\delta_1)+\Omega$ $(\delta_1\sinh\delta_2\cos\delta_1-\delta_2\cosh\delta_2\sin\delta_1)+2\beta^3\sinh\delta_2\sin\delta_1=0$ |
| C–G (or G–C) | $\delta_1\sinh\delta_2\cos\delta_1+\delta_2\cosh\delta_2\sin\delta_1-2\beta\sinh\delta_2\sin\delta_1=0$ |
| P–F (or F–P) | $2\beta\delta_1\delta_2\,(1-\cosh\delta_2\cos\delta_1)+\Omega$ $(\delta_1\sinh\delta_2\cos\delta_1-\delta_2\cosh\delta_2\sin\delta_1)+2\beta^3\sinh\delta_2\sin\delta_1=0$ |
| P–G (or P–G) | $\delta_1\delta_2\cosh\delta_2\cos\delta_1-\beta^2\sinh\delta_2\sin\delta_1=0$ |
| G–F (or F–G) | $\delta_1\sinh\delta_2\cos\delta_1+\delta_2\cosh\delta_2\sin\delta_1-2\beta\sinh\delta_2\sin\delta_1=0$ |

Remark: C: clamped; P: pinned; G: guided; F: free.

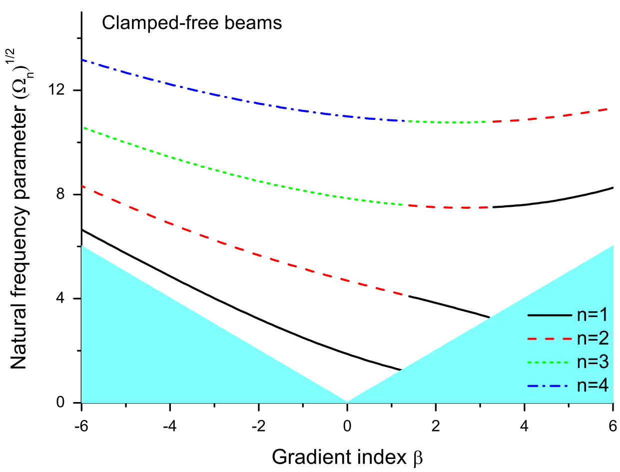


**Fig. 2.** Effects of the gradient parameter $\beta$ on the natural frequency parameter $\sqrt{\Omega_n}$ for cantilevered beams.

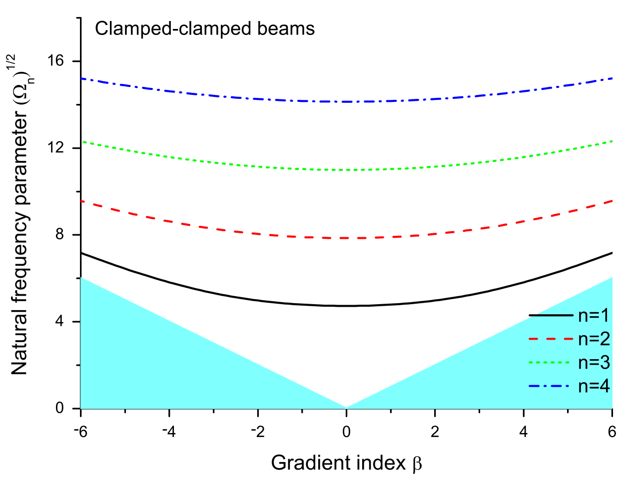


**Fig. 3.** Effects of the gradient parameter $\beta$ on the natural frequency parameter $\sqrt{\Omega_n}$ for clamped beams.

frequency parameter exhibits a jump behavior, the frequency parameter in other vibration modes also exhibits this feature, as seen in Fig. 2. When across certain $\beta$ values, the original $n$th natural frequency becomes the $(n-1)$th frequency. Such phenomena can

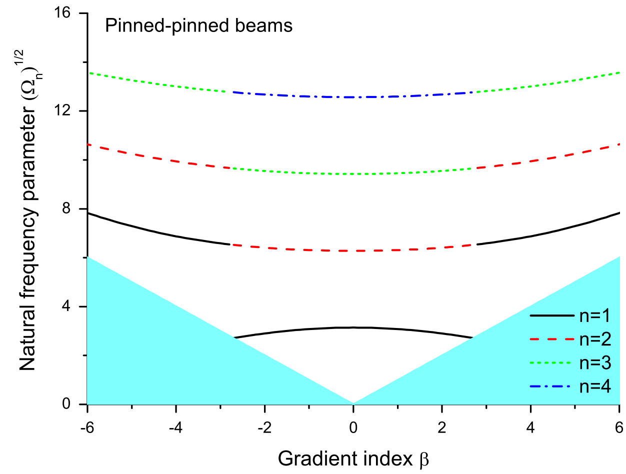


**Fig. 4.** Effects of the gradient parameter $\beta$ on the natural frequency parameter $\sqrt{\Omega_n}$ for simply-supported beams.

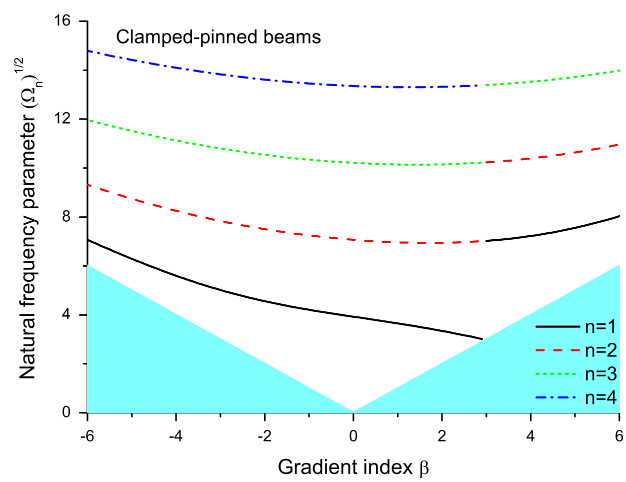


**Fig. 5.** Effects of the gradient parameter $\beta$ on the natural frequency parameter $\sqrt{\Omega_n}$ for clamped–pinned beams.

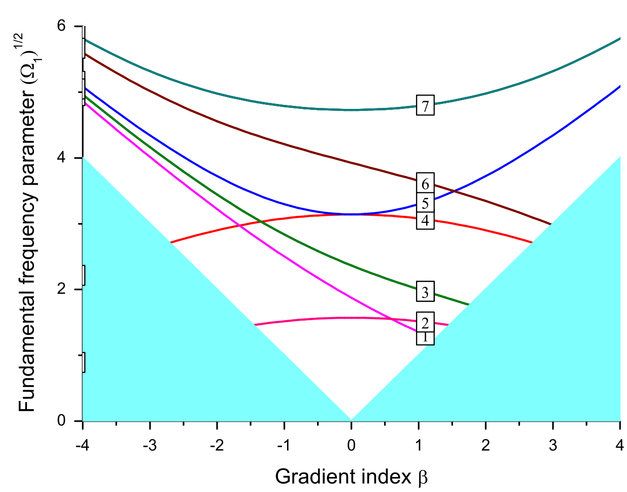


**Fig. 6.** The fundamental frequency parameter $\sqrt{\Omega_1}$ versus the gradient parameter $\beta$ for various beams: (1) Clamped–free beams. (2) Pinned–guided beams. (3) Clamped–guided or guided–free beams. (4) Pinned–pinned beams. (5) Guided–guided beams. (6) Clamped–pinned or pinned–free beams. (7) Clamped–clamped or free–free beams.

**Table 2**
Comparison of the dimensionless frequencies for a cantilever with exponential width variation.

| $-\beta$ | Present results | [25] | [23] | [24] | [27] | [22] | [21] |
|---|---|---|---|---|---|---|---|
| 0.5 | 4.73491 | 4.7349 | 4.72298 | 4.723 | 4.7347 | 4.735 | 4.5 |
| | 24.20181 | 24.202 | 24.20168 | 24.2017 | 24.2005 | 24.20 | 24.7 |
| | 63.86449 | 63.865 | 63.86448 | 63.8645 | 63.8608 | 63.85 | |
| | 123.09791 | 123.10 | 123.09790 | 123.098 | 123.091 | | |
| | 202.06877 | 202.07 | 202.06870 | | | | |
| 1 | 6.26264 | 2.2626 | 6.2588 | | | | |
| | 26.58359 | 26.584 | 26.584 | | | | |
| | 66.3745 | 66.375 | 66.374 | | | | |
| | 125.68472 | 125.69 | 125.68 | | | | |
| | 204.69531 | 204.70 | 204.70 | | | | |

**Table 3**
Dimensionless natural frequencies for a beam with exponential width variation, where those with asterisk specify pseudo-natural frequencies.

| $-\beta$ | CC | PP | CF | FC | CP | PC |
|---|---|---|---|---|---|---|
| 1 | 22.93773 | 9.48725 | 6.26264 | 1.84057 | 17.72026 | 13.34399 |
| | 62.42273 | 39.85232 | 26.58359 | 18.17212 | 52.52681 | 48.52769 |
| | 121.72273 | 89.40520 | 66.37450 | 58.38869 | 106.94600 | 102.94689 |
| | 200.71861 | 158.59689 | 125.68472 | 117.69217 | 181.03931 | 177.04168 |
| | 299.44014 | 247.48629 | 204.69531 | 196.70225 | 274.84490 | 270.84708 |
| 2 | 24.78955 | 8.41048 | 10.38723 | 0.90540* | 20.77798 | 11.18278 |
| | 64.70943 | 41.07056 | 32.15518 | 14.69432 | 56.29444 | 48.26067 |
| | 124.19583 | 91.17958 | 72.44786 | 56.50336 | 111.06221 | 103.07492 |
| | 203.30351 | 160.66459 | 132.03276 | 116.09870 | 185.35554 | 177.37515 |
| | 302.09813 | 249.73472 | 211.19719 | 195.25326 | 279.28915 | 271.30688 |
| 3 | 28.29800 | 6.84305* | 16.12268 | 0.42329* | 25.17737 | 8.87227* |
| | 68.63196 | 43.41110 | 39.03758 | 11.43261 | 61.42695 | 49.36407 |
| | 128.36481 | 94.25763 | 79.98808 | 56.21351 | 116.65731 | 104.71761 |
| | 207.63541 | 164.16870 | 139.96462 | 116.20441 | 191.24525 | 179.31557 |
| | 306.54165 | 253.51409 | 219.36758 | 195.56056 | 285.37673 | 273.43795 |

be explained as follows. For example, for $\beta = 5$, since $\delta_1$ in $\mathbf{V}$ in (21) is not real, but imaginary for $\sqrt{\Omega} < 5$, and so the fundamental frequency must be sought in $\sqrt{\Omega} > 5$. Therefore, in Fig. 2 the frequency parameter curves are seen to jump or discontinuous. For those falling in $\sqrt{\Omega} < \beta$, the beam does not give rise to harmonic vibration, which occurs only for $\sqrt{\Omega} > \beta$. This phenomenon is analogous to the second frequency spectrum of Timoshenko beams, which only exhibits the behavior of harmonic vibration for the frequency in excess of the corresponding critical frequency and possesses a nonpropagating field for the frequency below the corresponding critical frequency [26]. However, the first frequency spectrum of Timoshenko beams and the unique frequency spectrum of Euler–Bernoulli beams always correspond to the behavior of harmonic vibration. Here, for inhomogeneous Euler–Bernoulli beams only having the unique frequency spectrum, the phenomenon that Euler–Bernoulli beams have a nonpropagating field was not reported before. From Fig. 2, it is also viewed that for exponentially graded beams, we can find an optimal gradient index such that the fundamental frequencies reach minimum.

For other several typical cases, Figs. 3–5 display the effects of the gradient index $\beta$ on the natural frequency parameter $\sqrt{\Omega}$. It is easily viewed that the frequency parameter curves for clamped–clamped and pinned–pinned beams in Figs. 3 and 4 are symmetric. This is readily understood since the boundary conditions of two ends are the same for the two cases. Moreover, for clamped–clamped beams, the frequency curves are continuous, not discontinuous. For pinned–pinned beams and clamped–pinned beams, the frequency curves with a jump behavior still appear, similar to that observed in Fig. 2. A sole difference is that there is only a position at which the frequency jumps for pinned–pinned beams and clamped–pinned beams, while there are two jump locations for clamped–free beams. For other end supports of interest to us, the fundamental frequency parameter $\sqrt{\Omega}$ as a function of the gradient index $\beta$ is illustrated in Fig. 6. The frequency jump phenomenon occurs in most cases, and the frequency curves after jump are not depicted in Fig. 6.

We calculate the normalized natural frequencies of a cantilever beam with exponential bending stiffness and mass density, which corresponds a nonuniform cantilever beam with constant thickness and exponentially decaying width. For the latter case, exact results were derived by some researchers. A comparison of the normalized natural frequencies is made in Table 2. Obviously, it is found that our results are in agreement with existing natural frequencies. We also evaluate natural frequencies for four representative cases: clamped beams, simply-supported beams, cantilever beams, and clamped–pinned beams, and the obtained results are tabulated in Table 3 for several different values of $\beta$. Note that the beams with CF and FC boundary conditions in Table 3 stand for cantilever beams, where the first letter corresponds to the condition at the end $x = 0$ and the second to the end $x = L$, respectively. We see that for the PP beams, FC beams and PC beams with $\beta = 3$, the first characteristic value is less than $\beta^2$, and they are given in Table 3 with asterisk. In Figs. 2, 4 and 5, the corresponding frequency parameter curves fall in the shadow region, and associated vibration modes are dominated by (16). Since they correspond to nonpropagating fields, the natural frequencies are pseudo-natural frequencies. If neglecting these pseudo-natural frequencies, the fundamental frequencies are given by the second characteristic value or the first characteristic value larger than $\beta^2$.

Furthermore, the first four mode shapes of exponentially graded beams are presented in Fig. 7. Here for convenience, the mode shapes in (17) multiplying by the factor $e^{\beta x/L}$ have been plotted in these figures. Moreover, we do not normalize the mode shapes, but normalize the characteristic vector $\mathbf{A} = [A_1, A_2, A_3, A_4]$ with

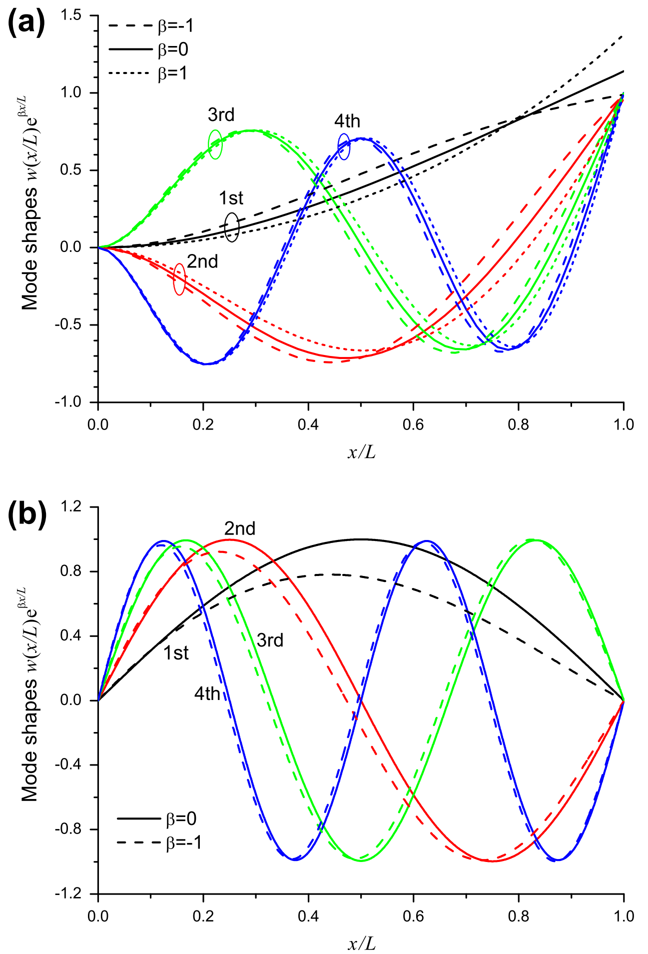


**Fig. 7.** The first four mode shapes, $W(x/L)e^{\beta x/L}$, versus $x/L$ for exponentially graded beams with $\beta = 1, 0, -1$: (a) clamped–free beams and (b) pinned–pinned beams.

$\|\mathbf{A}\| = 1$. It is observed from Fig. 7 that the material gradient has a strong influence on the first mode shape and the effect becomes smaller for higher-order vibration modes. It implies that the non-propagating components of bending waves corresponding to the terms related to the hyperbolic functions in (17) play a significant role in determining the amplitude of the lower-order vibration mode, while their effects are gradually weak with the mode number increasing. Due to this reason, with an increase in $\beta$, the effects of the propagating components of bending waves gradually become smaller and smaller, and finally the propagating components convert to nonpropagating components. Thus it gives rise to occurrence of the jump of the frequencies across which the trigonometric functions in (17) are guaranteed to avoid the appearance of the hyperbolic functions. This is also observed from Fig. 8 which displays the first two mode shapes of exponentially graded cantilever beams for three different values of $\beta = 0, 2, 5$. Notice that for $\beta = 2$, since the original fundamental frequency $\sqrt{\Omega_1} < 2$, which causes a non-harmonic vibration, the fundamental frequency then jumps up to the second frequency curve, as seen in Fig. 2, and the corresponding vibration shapes are dominated by propagating waves or harmonic vibration. For $\beta = 5$, since the fundamental frequency jumps twice, the vibration shape in mode 1 in Fig. 8c is seen to be similar to that in mode 3 in Fig. 7a. Consequently, thanks to this reason, the vibration shapes of mode 1 for $\beta = 0, 2, 5$ in fact are completely different, as viewed in Fig. 8. In contrast, the vibration shape of mode 1 for $\beta = 2$ is similar to that of mode 2 for $\beta = 0$, while the vibration shape of mode 1 for $\beta = 5$ is similar to that of

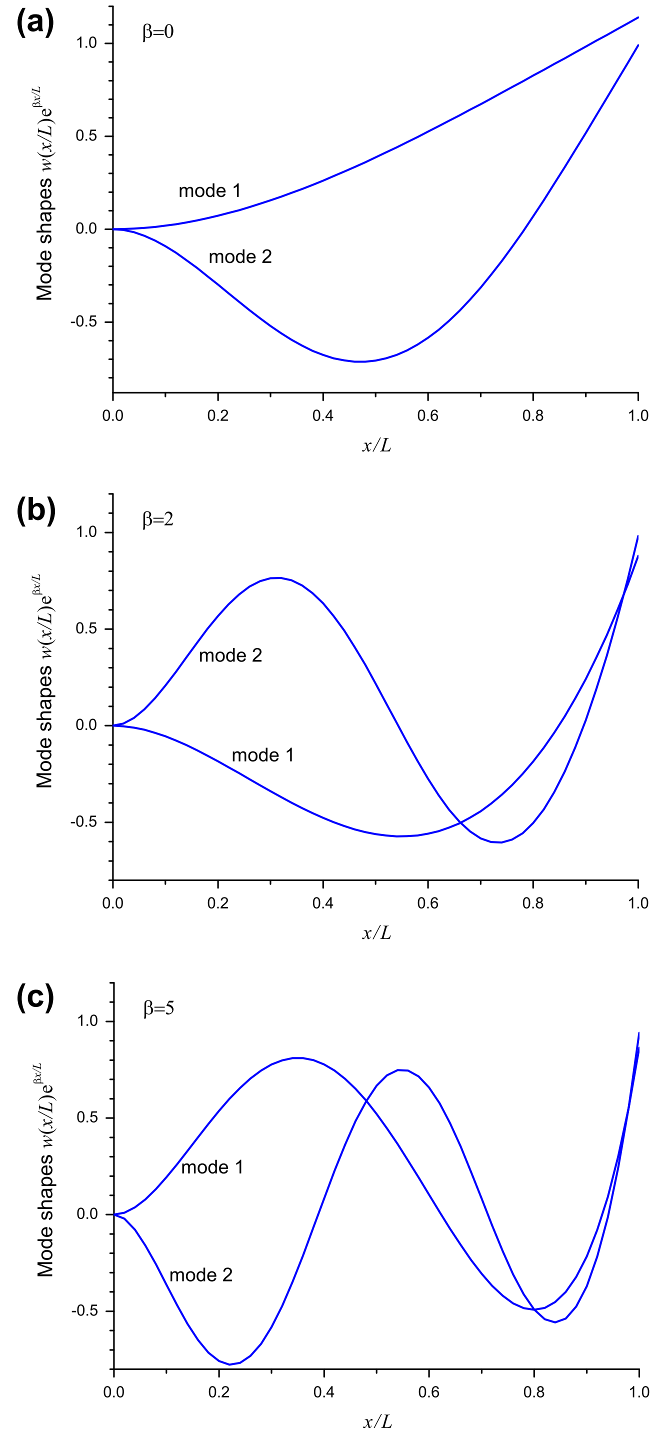


**Fig. 8.** The first two mode shapes, $W(x/L)e^{\beta x/L}$, versus $x/L$ for exponentially graded cantilevers with: (a) $\beta = 0$, (b) $\beta = 2$, and (c) $\beta = 5$.

mode 3 for $\beta = 0$ and of mode 2 for $\beta = 2$. This is attributed to the occurrence of the frequency jump phenomenon.

## 6. Conclusions

The free vibration of axially functionally graded beams was studied. For axially exponentially graded beams, closed-form characteristic equations have been derived for frequently-encountered end conditions. These characteristic equations reduce to the well-known characteristic equations if the gradient disappears. In particular, it is found that for certain graded beams, the natural frequencies jump when across its critical value and the natural frequencies lower than the critical value become pseudo-frequencies. Only when bending waves with frequencies exceeding the critical value, harmonic vibration can be excited. This is an essential difference between homogeneous and inhomogeneous beams. For certain inhomogeneous beams, this feature helps us to seek a minimal natural frequency for achieving optimal design of exponentially graded beams. All the results obtained apply to nonuniform beams with constant thickness and exponentially decaying width. The results of the present paper may serve as a benchmark for other numerical procedures for analyzing free vibration of axially functionally graded beams as well as nonuniform beams.

## References


[1] Sankar BV. An elasticity solution for functionally graded beams. Compos Sci Technol 2001;61:689–96.
[2] Zhong Z, Yu T. Analytical solution of a cantilever functionally graded beam. Compos Sci Technol 2007;67:481–8.
[3] Ding HJ, Huang DJ, Chen WQ. Elasticity solutions for plane anisotropic functionally graded beams. Int J Solids Struct 2007;44:176–96.
[4] Lu C, Chen W, Xu R, Lim C. Semi-analytical elasticity solutions for bi-directional functionally graded beams. Int J Solids Struct 2007:258–75.
[5] Kang Y-A, Li X-F. Bending of functionally graded cantilever beam with power-law non-linearity subjected to an end force. Int J Nonlinear Mech 2009;44: 696–703.
[6] Kang YA, Li XF. Large deflections of a non-linear cantilever functionally graded beam. J Reinf Plast Compos 2010;29:1761–74.
[7] Aydogdu M, Taskin V. Free vibration analysis of functionally graded beams with simply supported edges. Mater Design 2007;28:1651–6.
[8] Li X-F. A unified approach for analyzing static and dynamic behaviors of functionally graded Timoshenko and Euler–Bernoulli beams. J Sound Vib 2008;318:1210–29.
[9] Sina S, Navazi H, Haddadpour H. An analytical method for free vibration analysis of functionally graded beams. Mater Design 2009;30:741–7.
[10] Simsek M. Fundamental frequency analysis of functionally graded beams by using different higher-order beam theories. Nucl Eng Design 2010;240: 697–705.
[11] Huang Y, Li X-F. Bending and vibration of circular cylindrical beams with arbitrary radial nonhomogeneity. Int J Mech Sci 2010;52:595–601.
[12] Kang J, Leissa AW. Three-dimensional vibration analysis of thick, tapered rods and beams with circular cross-section. Int J Mech Sci 2004;46:929–44.
[13] Elishakoff I. Eigenvalues of inhomogenous structures: unusual closed-form solutions. New York: CRC Press; 2005.
[14] Huang Y, Li X-F. A new approach for free vibration of axially functionally graded beams with non-uniform cross-section. J Sound Vib 2010;329: 2291–303.
[15] Murin J, Aminbaghai M, Kutiš V. Exact solution of the bending vibration problem of FGM beams with variation of material properties. Eng Struct 2010;32:1631–40.
[16] Shahba A, Attarnejad R, Marvi MT, Hajilar S. Free vibration and stability analysis of axially functionally graded tapered Timoshenko beams with classical and non-classical boundary conditions. Compos Part B Eng 2011;42:801–8.
[17] Li XF, Xi LY, Huang Y. Stability analysis of composite columns and parameter optimization against buckling. Compos Part B Eng 2011;42:1337–45.
[18] Chakrabarti A, Ray PC, Bera RK. Large amplitude free vibration of a rotating nonhomogeneous beam with nonlinear spring and mass system. J Vib Acoust 2010;132:054502.
[19] Gilat R, Calio I, Elishakoff I. Inhomogeneous beams possessing an exponential mode shape. Mech Res Commun 2010;37:417–26.
[20] Karnovsky IA, Lebed OI. Formulas for structural dynamics: tables, graphs and solutions. New York: McGraw-Hill; 2004.
[21] Suppiger EW, Taleb NJ. Free lateral vibration of beams of variable cross section. ZAMP 1956;7:501–20.
[22] Cranch ET, Adler AA. Bending vibrations of variable section beams. J Appl Mech 1956;23:103–8.
[23] Ece MC, Aydogdu M, Taskin V. Vibration of a variable cross-section beam. Mech Res Commun 2007;34:78–84.
[24] Atmane HA, Tounsi A, Meftah SA, Belhadj HA. Free vibration behavior of exponential functionally graded beams with varying cross-section. J Vib Control 2011;17:311–8.
[25] Wang CY, Wang CM. Exact vibration solution for exponentially tapered cantilever with tip mass. J Vib Acoust 2012;134:041012.
[26] Stephen NG. The second spectrum of Timoshenko beam theory – further assessment. J Sound Vib 2006;292:372–89.
[27] Tong X, Tabarrok B, Yeh KY. Vibration analysis of Timoshenko beams with non-homogeneity and varying cross-section. J Sound Vib 1995;186:821–35.